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Closed-form magnetostatic interaction energies and forces in magnetic force microscopy

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Abstract. The magnetostatic energies and forces derived from axisymmetric models appropriate for magnetic force microscopy (MFM) of superconductors are examined. For models with a semi-infinite sample, closed form representations are obtained for arbitrary probe height. Specific boundary value problems considered are appropriate for a vortex penetrating a type-II superconductor, or for a magnetic monopole or dipole above or within a superconductor. Physically important limits such as complete flux expulsion become transparent with the new results. It is shown that previously employed approximations and numerical quadrature are unnecessary.

Introduction

Mathematical models of the interaction of stray magnetic fields with a magnetic probe tip are fundamental to the theory of magnetic force microscopy (MFM). Typical idealized tips are magnetic point charges (monopoles), magnetic point dipoles, and extended dipoles in the form of long needles, truncated cones and prisms, cylinders, and cylinders topped with a hemisphere or cone [1–4]. Recently, the magnetic force imaging of type-II superconductors, especially the high-T_c materials, has received attention. The MFM interaction in both the Meissner and vortex states has been considered. MFM provides new opportunities for studying the superconductor response in changes to parameters such as the temperature, magnetic field, and applied current density. Furthermore, future MFM studies may shed important information on the microscopic mechanisms of vortex pinning. This knowledge, in turn, will be very useful in understanding vortex dynamics and critical currents.

In superconductivity applications, MFM models typically involve a semi-infinite sample with either Abrikosov vortices or with magnetic point sources [1, 4]. The axisymmetry of the problem can be exploited. I have found that in such models the magnetostatic interaction energy and associated force can be put into closed form for arbitrary probe height. For models with a vertical vortex, all tip heights $z > 0$ directly above the vortex allow closed form results.

Formerly these energies and forces have been left in the form of one-dimensional integrals and evaluated numerically. I show that this procedure can be avoided. In this paper I present several classes of integrals which occur in MFM applications and give examples of their use. These techniques should also apply to magnetic levitation modelling [5–6] and MFM studies of recording media [3], but I concentrate on MFM studies of superconductors. I expect the results to be immediately useful alternatives to numerical quadrature.

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Due to the axisymmetry, the magnetostatic energies and forces are evaluated in terms of various solutions of the Bessel equation. The homogeneous solutions N_ν (or Y_ν) and K_ν appear, as do the inhomogeneous solutions $\boldsymbol{H}_\nu$ and $\boldsymbol{E}_\nu$ (the Struve and Weber functions). These results are currently all the more convenient because symbolic manipulation packages (SMPs) such as *Mathematica* and *Macsyma* [8] either have these functions built-in or are readily programmed. These SMPs usually have capabilities for evaluating recursion relations, taking derivatives, and plotting.

I begin with some specific integrals of interest and then show their applicability in several examples. There is little doubt that the MFM and magnetic and electric levitation literature [3–7] contains many more uses.

Reference integrals

I wish to focus on integrands containing products of algebraic functions and a decreasing exponential. The definite integral [9]

$$\int_0^\infty (x^2+a^2)^{1/2} x^n e^{-\mu x}\,dx = \frac{\pi}{2} a(-1)^n \frac{d^n}{d\mu^n}\frac{1}{\mu}[\boldsymbol{H}_1(a\mu) - N_1(a\mu)] \qquad \mathrm{Re}\,\mu > 0 \tag{1}$$

with integer n is very suitable. Here $\boldsymbol{H}_1$ and N_1 are the first-order Struve and Neumann functions. The extension to $n \neq 0$ is important in the following. When the square root instead appears in the denominator I have [9]

$$\int_0^\infty (x^2+a^2)^{-1/2} x^n e^{-\mu x}\,dx = \frac{\pi}{2}(-1)^n \frac{d^n}{d\mu^n}[\boldsymbol{H}_0(a\mu) - N_0(a\mu)] \qquad \mathrm{Re}\,\mu > 0\,. \tag{2}$$

A simple connection between (1) and (2) is that $\partial/\partial a$ of the integral of (1) is a times the integral of (2). Similarly, the closely related definite integral with power n

$$\int_0^\infty \frac{e^{-\mu x} x^n}{\sqrt{x^2+1}+1}\,dx = (-1)^n \frac{d^n}{d\mu^n}\left\{\frac{\pi}{2}\frac{1}{\mu}[\boldsymbol{H}_1(\mu) - N_1(\mu)] - \frac{1}{\mu^2}\right\} \qquad \mathrm{Re}\,\mu > 0 \tag{3}$$

is highly applicable. Equation (3) also follows from (1) and the last term of (3) is simply $\Gamma(n+2)/\mu^{n+2}$. The utility of (1)–(3) is underscored by the recursion relations satisfied by the Struve and Neumann functions. In particular, $\boldsymbol{H}_0'(z) = 2/\pi - \boldsymbol{H}_1(z)$, $N_0'(z) = -N_1(z)$, and

$$\frac{d}{dz}\frac{1}{z}\boldsymbol{H}_1(z) = \frac{2}{3\pi} - \frac{1}{z}\boldsymbol{H}_2(z) \tag{4a}$$

$$\frac{d}{dz}\frac{1}{z}N_1(z) = -\frac{1}{z}N_2(z) \tag{4b}$$

show that the derivatives are easy to compute. By way of the recursion relations, the higher-order derivatives of the Struve and Neumann functions in (1)–(3) can be written in terms of these functions themselves. In this way, as illustrated below, I am able to avoid using a Meijer function G^{31}_{13} [9], where

$$\int_0^\infty (x^2+a^2)^{1/2} x^n e^{-\mu x}\,dx = -\frac{a^{n+2}}{4\pi} G^{31}_{13}\left(\frac{\mu^2 a^2}{4}\,\middle|\,{}^{1/2-n/2}_{-n/2-1,0,1/2}\right) \qquad \mathrm{Re}\,\mu > 0\,. \tag{5}$$

That a Meijer function G^{31}_{13} is related to a Lommel function $S_{\mu,\nu}$ is not surprising. Another means of evaluating (1) in terms of Weber and Neumann functions is given in the appendix.

Lastly I require the integral representations of the Bessel functions. Only one of many equivalent forms is given here:

$$\int_1^\infty (x^2-1)^\nu \mathrm{e}^{-\mu x}\,\mathrm{d}x = \frac{1}{\sqrt{\pi}}\left(\frac{2}{\mu}\right)^{\nu+1/2}\Gamma(\nu+1)K_{\nu+1/2}(\mu) \qquad \mathrm{Re}\,\mu>0 \quad \mathrm{Re}\,\nu>-1\,. \tag{6}$$

Note that half-integer-order modified Bessel functions are simply products of elementary functions.

Examples for semi-infinite superconductor geometry

As a first elementary example, consider a type-II superconductor occupying the half-space $z<0$ and containing a single vortex oriented along the z-direction [1, 10, 11]. The maximum value of the vertical component of the magnetic field above the superconductor is given by

$$H_z(\rho=0,z\geqslant 0) = \frac{\phi_0}{2\pi\lambda^2}\int_0^\infty \frac{\mathrm{d}k\,k}{\sqrt{1+k^2}}\frac{\mathrm{e}^{-kz/\lambda}}{(k+\sqrt{1+k^2})} \tag{7}$$

where λ is the London penetration depth and ϕ_0 is the flux quantum. This integral can be evaluated by decomposing the integrand and using (2) and (3). The result for the maximum value as a function of separation z is

$$H_z(0,z\geqslant 0) = \frac{\phi_0}{4\lambda^2}\left[\boldsymbol{H}_0\left(\frac{z}{\lambda}\right) - \frac{\lambda}{z}\boldsymbol{H}_1\left(\frac{z}{\lambda}\right) - N_0\left(\frac{z}{\lambda}\right) + \frac{\lambda}{z}N_1\left(\frac{z}{\lambda}\right) + \frac{2}{\pi}\frac{\lambda^2}{z^2}\right]. \tag{8}$$

Here [9]

$$\boldsymbol{H}_0(z) = \frac{2}{\pi}\int_0^{\pi/2}\sin(z\cos\varphi)\,\mathrm{d}\varphi \tag{9}$$

is $-\boldsymbol{E}_0(z)$ where $\boldsymbol{E}_0$ is the zero-order Weber function, and $\boldsymbol{H}_1(z)=2/\pi-\boldsymbol{H}_0'(z)$. In figure 1 I have plotted $(4\lambda^2/\phi_0)H_{1z}(\rho=0,z)$ versus z/λ. The maximum value of the vertical component of the magnetic field is monotonically decreasing with separation z, as expected physically. It is remarkable that the separately oscillating Struve and Bessel functions in (8) yield such a decreasing function. In some MFM tip models, such as the magnetic point-charge model [1], H_z yields the vertical force F_z directly. Therefore (8) quantitatively gives the expected size of the force signal due to a single vortex when $F_z\propto H_z$.

Next consider a magnetic point charge, m, at a distance d above a semi-infinite superconductor [1]. The vertical component of the induced magnetic induction is

$$B_z^{ind}(\rho,z>0) = \frac{m}{4\pi}\int_0^\infty \frac{\sqrt{k^2+\lambda^{-2}}-k}{\sqrt{k^2+\lambda^{-2}}+k}\mathrm{e}^{-k(d+z)}J_0(k\rho)k\,\mathrm{d}k\,. \tag{10}$$

Here J_0 is the zero-order Bessel function and ρ is the radial coordinate. The force acting on the tip is $F_z=m_zH_z=\frac{m}{\mu_0}B_z^{ind}(\rho=0,z=d)$. The resulting integral for the force is performed by first rationalizing the denominator of the integrand. Then two terms may be integrated trivially, giving

$$F_z(d) = \frac{m^2}{4\pi\mu_0}\frac{\lambda^2}{d^2}\left[\frac{1}{4\lambda^2}+\frac{3}{4d^2}-2d^2\int_0^\infty k^2\sqrt{k^2+\lambda^{-2}}\mathrm{e}^{-2kd}\,\mathrm{d}k\right]. \tag{11}$$

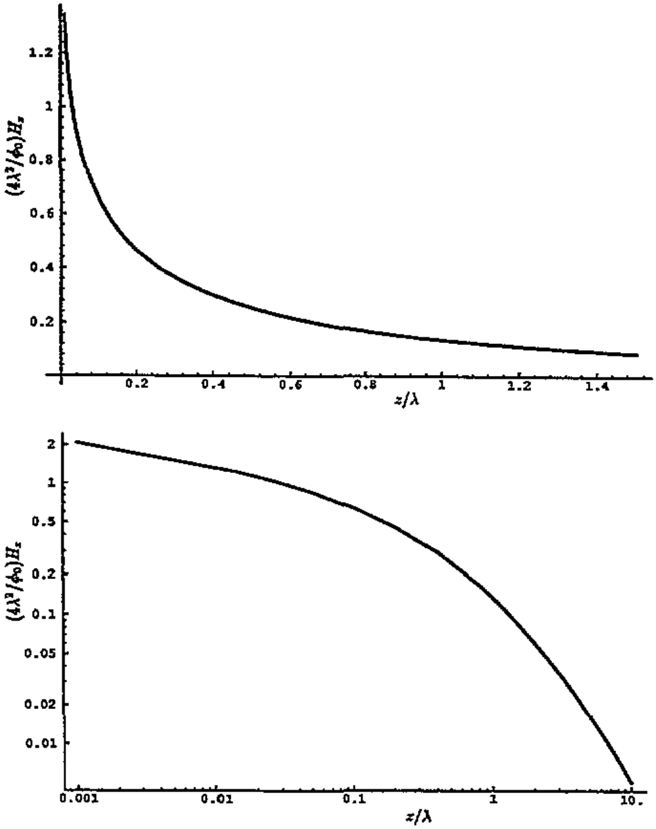

Figure 1. (*a*) Linear plot of the maximum value of the vertical component of magnetic field $(4\lambda^2/\phi_0)H_z(\rho = 0, z)$, above a semi-infinite superconductor (equation (8)) versus z/λ, where λ is the London penetration depth. (*b*) Log–log plot of $(4\lambda^2/\phi_0)H_z$ versus z/λ.

The remaining integral is evaluated by using (1) and (4), and the relations

$$\frac{1}{z}N_2' = \frac{1}{z}N_1 - \frac{2}{z^2}N_2 \tag{12a}$$

$$\frac{1}{z}\boldsymbol{H}_2' = \frac{1}{z}\boldsymbol{H}_1 - \frac{2}{z^2}\boldsymbol{H}_2. \tag{12b}$$

The resulting expression for the vertical force is

$$\frac{4\pi\mu_0}{m^2}\lambda^2 F_z = \frac{\lambda^2}{4d^2} + \frac{3}{4}\frac{\lambda^4}{d^4} - \pi\frac{\lambda}{d}\left[\frac{3}{4}\frac{\lambda}{d}\boldsymbol{H}_2\left(\frac{2d}{\lambda}\right) - \frac{1}{2}\boldsymbol{H}_1\left(\frac{2d}{\lambda}\right)\right.$$
$$\left. - \frac{3}{4}\frac{\lambda}{d}N_2\left(\frac{2d}{\lambda}\right) + \frac{1}{2}N_1\left(\frac{2d}{\lambda}\right)\right]. \tag{13}$$

The expected scaling dependence of the force is explicitly manifest. In figure 2, $\frac{4\pi\mu_0}{m^2}\lambda^2 F_z$ is plotted versus distance d/λ (log–log scale). The curves for $\lambda^2/4d^2$ and $\lambda^2/4(d+1)^2$ are included for comparison. The function $1/4d^2$ gives a strict upper bound on the force, corresponding to complete flux exclusion ($\lambda \to 0$).

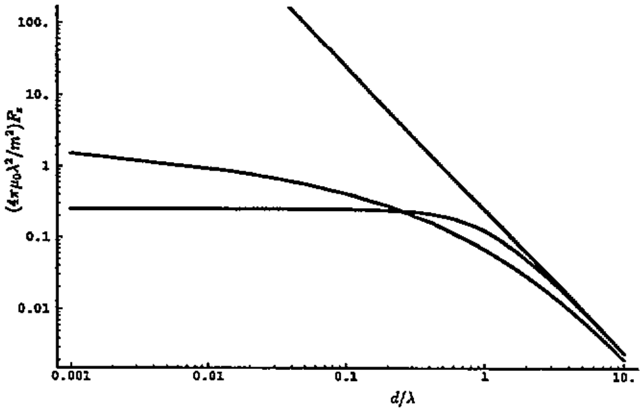

Figure 2. Plot of the vertical force on a magnetic point charge (equation (13)), $\frac{4\pi\mu_0}{m^2}\lambda^2 F_z$ versus distance d/λ (log–log scale). The curves for $\lambda^2/4d^2$ and $\lambda^2/4(d+1)^2$ are included for comparison.

In a similar way the induced magnetic induction itself can be evaluated in closed form. Using the definite integral [9]

$$\int_0^\infty x^n e^{-\alpha x} J_0(\beta x)\,dx = (-1)^n \frac{d^n}{d\alpha^n}\frac{1}{\sqrt{\alpha^2+\beta^2}} \qquad \mathrm{Re}\,\alpha > 0 \tag{14}$$

gives

$$B_z^{ind}(\rho, z) = \frac{m}{4\pi}\frac{(d+z)}{[(d+z)^2+\rho^2]^{3/2}}\left\{1+\frac{\lambda^2[2(d+z)^2-3\rho^2]}{[(d+z)^2+\rho^2]^2}\right\} - \frac{m\lambda^2}{2\pi}\int_0^\infty k^2\sqrt{k^2+\lambda^{-2}}e^{-k(d+z)}J_0(k\rho)\,dk \qquad z > 0. \tag{15}$$

The last term in (15), $-(m/2\pi\lambda^2)\int_0^\infty k^2\sqrt{k^2+1}e^{-\alpha k/\lambda}J_0(k\rho/\lambda)$, $\alpha = d+z$, can be evaluated by either expanding J_0 or the square root in infinite series. Using the infinite series form of J_0 and (1) gives for this term

$$-\frac{m}{4\lambda^2}\sum_{l=0}^{\infty}\frac{(-1)^l}{2^{2l}(l!)^2}\left(\frac{\rho}{\lambda}\right)^{2l}\frac{d^{2l+2}}{d\mu^{2l+2}}\frac{1}{\mu}[H_1(a\mu)-N_1(a\mu)] \tag{16}$$

with $\mu = \alpha/\lambda$. Otherwise it can be rewritten as

$$-\frac{m}{2\pi\lambda^2}\sum_{l=0}^{\infty}\binom{\frac{1}{2}}{l}\left(\frac{d}{d\mu}\right)^{2l+2}\frac{1}{(\mu^2+\rho^2/\lambda^2)^{1/2}} \tag{17a}$$

where the binomial coefficient is

$$\binom{\frac{1}{2}}{l} = \frac{(2l-1)!!}{(\frac{1}{2}-l)l!(-1)^l 2^{l+1}}. \tag{17b}$$

Similar closed form results can be obtained for the supercurrent density in the superconductor ($z < 0$) and for the vector potential and both components of the magnetic field in all space, but these are not primary to the present discussion.

The remaining examples of this section concern a magnetic point dipole located either above or within a semi-infinite superconductor [4]. The next section treats a model of an extended MFM probe, namely a vertical segment of wire. If the point dipole is located a distance a above the superconductor, with angle of inclination θ with respect to the vertical (z) axis, then the magnetostatic interaction energy is [4]

$$U(a) = (1+\cos^2\theta)\frac{\mu_0 m^2}{16\pi\lambda^3}\int_0^\infty (1+2k^2-2k\sqrt{1+k^2})k^2 e^{-2ka/\lambda}\,dk \qquad a>0. \tag{18}$$

This energy has been obtained from [4] $U(a) = -\frac{1}{2}\vec{m}\cdot\vec{B}^{ind}(\rho=0, z=a)$ where $\vec{m}$ is the dipole moment and the factor $\frac{1}{2}$ is due to self-interaction. Integrating the first two terms of (18) leaves

$$U(a) = (1+\cos^2\theta)\frac{\mu_0 m^2}{64\pi a^3}\left[1+6\frac{\lambda^2}{a^2}-8\frac{a^3}{\lambda^3}\int_0^\infty \sqrt{1+k^2}k^3 e^{-2ka/\lambda}\,dk\right] \tag{19}$$

where the integral differs from that in (11) only in the additional power of k. Therefore equation (1) gives

$$U(a) = (1+\cos^2\theta)\frac{\mu_0 m^2}{64\pi a^3}\left\{1+6\frac{\lambda^2}{a^2}+2\pi\frac{a^2}{\lambda^2}\left[\left(-3\frac{\lambda^2}{a^2}+1\right)H_2\left(\frac{2a}{\lambda}\right)+\frac{3}{2}\frac{\lambda}{a}H_1\left(\frac{2a}{\lambda}\right)\right.\right.$$
$$\left.\left.-\frac{4}{3\pi}\frac{a}{\lambda}+\left(3\frac{\lambda^2}{a^2}-1\right)N_2\left(\frac{2a}{\lambda}\right)-\frac{3}{2}\frac{\lambda}{a}N_1\left(\frac{2a}{\lambda}\right)\right]\right\}. \tag{20}$$

This equation shows that both the interaction energy and vertical force

$$F_z = -\frac{\partial U}{\partial a} = (1+\cos^2\theta)\frac{\mu_0 m^2}{8\pi\lambda^4}\int_0^\infty (1+2k^2-2k\sqrt{1+k^2})k^3 e^{-2ka/\lambda}\,dk \tag{21}$$

can be put into closed form for arbitrary dipole orientation. In figure 3, for $\theta = 0$, $(8\pi\lambda^3/\mu_0 m^2)U(a)$ is plotted versus the normalized height a/λ (log–log scale).

For a vertical dipole located within the semi-infinite superconductor, at a distance a from the surface, the magnetostatic interaction energy is given by [4]

$$-\frac{8\pi\lambda^3}{\mu_0 m^2}U_{ver}(a) = \int_0^\infty \frac{k^3}{\sqrt{1+k^2}}(1+2k^2-2k\sqrt{1+k^2})e^{-2a\sqrt{1+k^2}/\lambda}\,dk \tag{22}$$

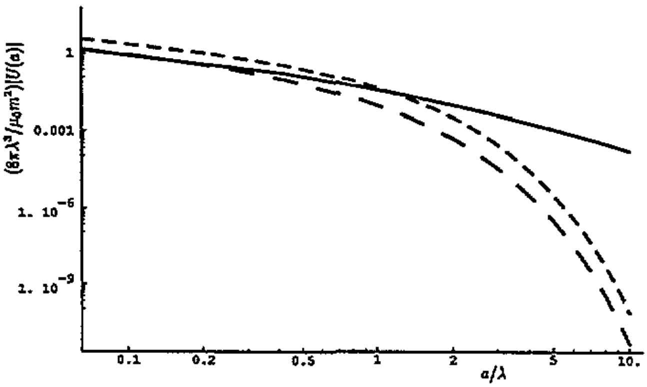

Figure 3. Plot of the magnetostatic interaction energies, equations (20), (24) and (27), for a dipole located a distance a either above or below the surface of a semi-infinite superconductor. On a log–log scale $(8\pi\lambda^3/\mu_0 m^2)|U(a)|$ is plotted versus a/λ, where the dipole moment is m and λ is the London penetration depth. For equation (20), the full curve is for $\theta = 0$, the short-broken curve is for $U_{ver}(a)$, and the long-broken curve is for $U_{hor}(a)$.

where the presence of the radical in the exponent is due to the finite penetration depth. From equation (6) it follows that

$$\int_1^\infty (x^2-1)^\nu e^{-\mu x} x\, dx = \frac{1}{\sqrt{\pi}}\left(\frac{2}{\mu}\right)^{\nu+1/2} \Gamma(\nu+1) K_{\nu+3/2}(\mu) \qquad \mathrm{Re}\,\mu > 0 \quad \mathrm{Re}\,\nu > -1 \tag{23}$$

and then (22) may be evaluated entirely in terms of modified Bessel functions. The functions $K_{3/2}$ and $K_{5/2}$ can be rewritten in terms of elementary functions, giving, with $z = a/\lambda$,

$$-\frac{8\pi\lambda^3}{\mu_0 m^2} U_{ver}(a) = \frac{1}{2z^2}\left\{ e^{-2z}\left[1 + \frac{9}{2}\frac{1}{z} + \frac{6}{z^2} + \frac{3}{z^3}\right] - 3K_3(2z)\right\}. \tag{24}$$

This function is also plotted in figure 3.

On the other hand, when there is a horizontal dipole located within the superconductor, the interaction energy is [4]

$$-\frac{16\pi\lambda^3}{\mu_0 m^2} U_{hor}(a) = \int_0^\infty \frac{k[(2+k^2)\sqrt{1+k^2} - k^3]}{\sqrt{1+k^2}(\sqrt{1+k^2}+k)} e^{-2a\sqrt{1+k^2}/\lambda}\, dk. \tag{25}$$

The decomposition

$$-\frac{16\pi\lambda^3}{\mu_0 m^2} U_{hor}(a) = \int_0^\infty \left[2k(\sqrt{1+k^2}-k) + \frac{k^3+2k^5}{\sqrt{1+k^2}} - 2k^4\right] e^{-2a\sqrt{1+k^2}/\lambda}\, dk \tag{26}$$

is very useful because it can be recognized that only two additional integrals are introduced beyond (22). Of these, one is elementary and the other can be evaluated by using (23) with $\nu = \frac{1}{2}$. After combining like powers of a/λ and using a recursion relation for $K_3(2a/\lambda)$, this leads to

$$\begin{aligned} -\frac{16\pi\lambda^3}{\mu_0 m^2} U_{hor}(a) &= e^{-2a/\lambda}\frac{\lambda}{a}\left(1 + \frac{3}{2}\frac{\lambda}{a} + \frac{11}{4}\frac{\lambda^2}{a^2} + 3\frac{\lambda^3}{a^3} + \frac{3}{2}\frac{\lambda^4}{a^4}\right) \\ &\quad - \frac{\lambda}{a}\left[\frac{3\lambda}{2a}K_1\left(\frac{2a}{\lambda}\right) + \left(1 + \frac{3\lambda^2}{a^2}\right)K_2\left(\frac{2a}{\lambda}\right)\right]. \end{aligned} \tag{27}$$

In figure 3, $-(8\pi\lambda^3/\mu_0 m^2)U(a)$ is plotted versus the normalized depth a/λ (log–log scale). (In equation (27) K_2 could be written in terms of K_0 and K_1.)

By superposition, the magnetostatic interaction energy may be obtained as [4]

$$U(a) = U_{ver}(a)\cos^2\theta + U_{hor}(a)\sin^2\theta \tag{28}$$

for arbitrary dipole orientation within the superconductor. Using equation (24) for U_{ver} and (27) for U_{hor}, I have shown that the interaction energy and associated force may be put into closed form for any dipole direction. Furthermore, using (26), I have shown that

$$2U_{hor}(a) = -\frac{\lambda}{a}\frac{\mu_0 m^2}{8\pi\lambda^3}\left[e^{-2a/\lambda}\left(1 + \frac{\lambda}{a} + \frac{\lambda^2}{2a^2}\right) - K_2\left(\frac{2a}{\lambda}\right)\right] + U_{ver}(a). \tag{29}$$

By using the small argument expansion $K_2(x) \to 1/2x^2 - 1/2 + O(x^2)$ as $x \to 0$, it can be shown that $2U_{hor}$ and U_{ver} differ by a term proportional to $1/a$ as $a \to 0$. Similarly, it can be shown that both U_{ver} and U_{hor} vary as $1/a$ as $a \to 0$.

Straight wire tip for MFM

Here a straight-line wire tip of length l and magnetized along the $\hat{z}$-direction is considered. The wire is oriented vertically, with the bottom end at a distance of a from the superconductor. This provides a very simple model of a spatially extended MFM probe. Another reason it is of interest is because it contains both the monopole and dipole results as special cases. The self-interaction energy can be written as

$$U(a) = \frac{\mu_0 q^2}{8\pi\lambda}\int_0^\infty (1+2k^2-2k\sqrt{1+k^2})\mathrm{e}^{-2ka/\lambda}(1-\mathrm{e}^{-kl/\lambda})^2\,\mathrm{d}k \qquad a>0 \tag{30}$$

where q is the magnetic moment density per unit length. Then the force can be expressed as [4]

$$\frac{4\pi\lambda^2}{\mu_0 q^2}F_z(a) = \int_0^\infty (1+2k^2-2k\sqrt{1+k^2})k\mathrm{e}^{-2ka/\lambda}(1-\mathrm{e}^{-kl/\lambda})^2\,\mathrm{d}k\,. \tag{31}$$

Previously this integral was approximated and numerically evaluated [4]. Using equations (1), (4), and (12), the force follows as

$$\begin{aligned}\frac{4\pi\lambda^2}{\mu_0 q^2}F_z &= \lambda^2\left[\frac{1}{4a^2}-\frac{2}{(2a+l)^2}+\frac{1}{4(a+l)^2}\right]+4\lambda^3\left[\frac{1}{8a^3}-\frac{2}{(2a+l)^3}+\frac{1}{8(a+l)^3}\right]\\ &\quad -\pi\left\{\frac{\lambda}{2a}\left[\frac{3}{2}\frac{\lambda}{a}\boldsymbol{H}_2\left(\frac{2a}{\lambda}\right)-\boldsymbol{H}_1\left(\frac{2a}{\lambda}\right)-\frac{3}{2}\frac{\lambda}{a}N_2\left(\frac{2a}{\lambda}\right)+N_1\left(\frac{2a}{\lambda}\right)\right]\right.\\ &\quad -\frac{2\lambda}{2a+l}\left[\frac{3\lambda}{(2a+l)}\boldsymbol{H}_2\left(\frac{2a+l}{\lambda}\right)-\boldsymbol{H}_1\left(\frac{2a+l}{\lambda}\right)-\frac{3\lambda}{2a+l}N_2\left(\frac{2a+l}{\lambda}\right)\right.\\ &\quad \left.+N_1\left(\frac{2a+l}{\lambda}\right)\right]+\frac{\lambda}{2(a+l)}\left[\frac{3\lambda}{2(a+l)}\boldsymbol{H}_2\left(\frac{2(a+l)}{\lambda}\right)-\boldsymbol{H}_1\left(\frac{2(a+l)}{\lambda}\right)\right.\\ &\quad \left.\left.-\frac{3\lambda}{2(a+l)}N_2\left(\frac{2(a+l)}{\lambda}\right)+N_1\left(\frac{2(a+l)}{\lambda}\right)\right]\right\}.\end{aligned} \tag{32}$$

As expected, extreme lengths of the wire provide useful special cases. As $l \to 0$, the magnetic dipole limit is recovered, equation (21) with $\theta = 0$. The product lq is identified with the magnetic dipole moment m. As $l \to \infty$, the magnetic monopole result (equation (11)) is recovered. In this limit, the upper magnetic charge does not interact with the superconductor.

Discussion and summary

This paper examined the magnetostatic energies and forces derived from axisymmetric models appropriate for magnetic force microscopy (MFM) of superconductors. The geometry of such models typically has a semi-infinite sample with either an Abrikosov vortex and/or magnetic point sources. By means of a two-dimensional Fourier transformation, or Hankel transformation, these models can be solved. The resulting electromagnetic fields and densities and associated energies and forces have previously been left in the form of one-dimensional integrals over wavenumber k. This paper shows that these results can be made much more explicit.

Terms which correspond to physically important limits become transparent. An example is the limiting situation of complete flux expulsion, $\lambda \to 0$, as in (11), where λ is the London penetration depth. Even for the well studied Pearl solution [1, 10] new closed form results have been obtained.

This paper shows that several previous uses of numerical integration are unnecessary. A key is to use the special function theory of solutions of Bessel's equation, including the properties specific to the Bessel, modified Bessel, Neumann, Struve, and Weber functions. The use of the Meijer function [9] G_{13}^{31} has been avoided because a detailed knowledge of its transformation properties would be required. Although special functions appear in intermediate results, often elementary functions arise in special cases. An example is (6) when ν is an integer. Another point of view in considering this result is to think in terms of incomplete gamma functions [9] $\gamma(\alpha, x)$ or $\Gamma(\alpha, x)$.

A basic approach of this paper was to exploit differentiation of an integral with respect to a parameter, as in (1)–(3) and (14). This was particularly convenient for the integrals of interest because of the recursion relations satisfied by solutions of the homogeneous or inhomogeneous Bessel equation. For instance, the three-term recurrence relation for the derivatives of the Struve function $\boldsymbol{H}_\nu$ involves only these functions. This method has manifold additional applications in the solution of vortex and electromagnetic boundary value problems which in turn is useful for MFM and levitation modelling.

The results of this paper can be used in further quantitative MFM studies of superconductors to delineate the deviations from the idealizations. For instance, it is usually assumed that the probe itself does not disturb the field being imaged. Comparison with the results here could show how large the perturbing effect of the probe is. Also important for finite size probes could be to take into account demagnetization effects. The models treated here have not taken into account hysteretic magnetic forces, which depend on the history of the sample. Related to this topic is the need to accommodate for critical state effects of a type-II superconductor.

The results of this paper can be extended and applied to a variety of levitation systems [5–7]. Consider, for example, the hysteretic force on a small volume V of superconductor magnetized parallel to $\vec{H}$, $\vec{F} = \mu_0 V M(H) \nabla H$, where $M(H)$ is the magnetization curve. For a monopole field, as for (10)–(17), H varies as $1/z^2$ for $\rho = 0$ and large z. Then $\partial H/\partial z \sim 1/z^3$ along the vertical axis and the vertical levitation force is $F_z \sim V M(H) H^{3/2}$.

A wide class of stratified superconductor boundary value problems can be solved within the framework recently developed in [11]. These authors have shown how H_z can be used as a scalar potential and treated very general vorticities for the mixed state. This theory can also be used to model a wide range of magnetic force and levitation systems.

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Appendix. Reference integral

This appendix presents an alternative for evaluating integrals of the form

$$I_n(\beta) = \int_0^\infty k^n \sqrt{1+k^2} \mathrm{e}^{-\beta k} \,\mathrm{d}k \qquad \mathrm{Re}\,\beta > 0 \tag{A1}$$

where n is a non-negative integer, such as occur in (1), (11), (19), and (21). With the change of variable $k = \sinh t$, I have

$$I_n(\beta) = \int_0^\infty \sinh^n t \cosh^2 t \mathrm{e}^{-\beta \sinh t} \,\mathrm{d}t\,. \tag{A2}$$

When n is odd, it is convenient to write the factor $\sinh^n t + \sinh^{n+2} t$ as a sum of hyperbolic sines of odd multiples of t. Then appeal can be made to the integral [9]

$$\int_0^\infty \sinh \gamma x e^{-\beta \sinh x}\, dx = -\frac{\pi}{2}\left[E_\gamma(\beta) + N_\gamma(\beta)\right] \qquad \mathrm{Re}\,\beta > 0 \tag{A3}$$

where γ is an odd integer and E_γ is the Weber function of order γ. When n is even in (A2), it is useful to write the factor $(\cosh^2 t - 1)^n \cosh^2 t$ as a sum of hyperbolic cosines of even multiples of t. Then the integral [9]

$$\int_0^\infty \cosh \nu x e^{-\beta \sinh x}\, dx = -\frac{\pi}{2}\left[E_\nu(\beta) + N_\nu(\beta)\right] \qquad \mathrm{Re}\,\beta > 0 \tag{A4}$$

where ν is an even integer, can be applied.

Evaluation of the integral $I_n(\beta)$ can then be made in terms of lower-order Struve functions $H_p(\beta)$ by way of the recursion relation [9]

$$E_{\nu+1}(z) = -E_{\nu-1}(z) + \frac{2\nu}{z} E_\nu(z) \begin{cases} +0 & \text{for } \nu \text{ even} \\ -\dfrac{4}{\pi z} & \text{for } \nu \text{ odd} \end{cases} \tag{A5}$$

for the Weber functions and the relation [12]

$$E_n(z) = \frac{1}{\pi} \sum_{k=0}^{[(n-1)/2]} \frac{\Gamma(k+1/2)(z/2)^{n-2k-1}}{\Gamma(n+1/2-k)} - H_n(z) \tag{A6}$$

where $n \geqslant 0$ is an integer. The order of the Neumann functions N_ν is readily reduced with the use of

$$N_{\nu+1}(z) = -N_{\nu-1}(z) + \frac{2\nu}{z} N_\nu(z). \tag{A7}$$

In particular, in evaluating (11) and (19) the following relations are practical:

$$E_4(\beta) = \left(\frac{24}{\beta^2} - 1\right) E_2(\beta) - \frac{6}{\beta} E_1(\beta) - \frac{4}{\pi\beta} \tag{A8a}$$

$$E_5(\beta) = \frac{12}{\beta}\left(\frac{16}{\beta^2} - 1\right) E_2(\beta) + \left(1 - \frac{48}{\beta^2}\right) E_1(\beta) - \frac{32}{\pi\beta^2}. \tag{A8b}$$

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